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The Plastico-Viscous Deformation of Right Circular Cylinders of Soft Metal under Variable Load Axially Directed

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IV. *The Plastico-Viscous Deformation of Right Circular Cylinders of Soft Metal under Variable Load Axially Directed.*

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Introductory.

1. Soft metals are of service to man in many ways, extended use following each advance in knowledge. The present investigation had its origin in an attempt to account analytically for the many apparently irreconcilable properties exhibited by right circular cylinders of soft copper when subjected to appreciable strain under heavy crushing loads. During the course of the work it became clear that if advance at all were to be made general methods of analysis would require to be adopted; and that results forthcoming would be applicable to materials, of a similar nature, other than the one directly used as a standard of comparison. The presentation of the subject matter has accordingly been arranged to give prominence to theoretical and rational aspects, references to experiment being rather for comparison than for support to the argument. Two publications will be found of service in the present connection, the one illustrative of a cogent point in the theory, the other descriptive of practice. They will be referred to as I, and II,* respectively.

2. In practice no method is available for ascertaining the nature of the medium whose properties it is proposed to investigate while the strains are taking place. After-inspection and examination indicate that changes can be discerned consequent on the effect of applied stress and heat. In the subsequent analysis, as developed, the changes are taken to be after-effects, and during the straining the properties of the medium are presumed to remain unchanged. The experimental evidence in general leads to the view that the underlying and essential phenomenon is elastic in nature, that the effects of this property will be most readily discernible, if at all, at the start and end of motion, possibly also at free boundaries, and that during plastic motion the elastic strains,

* I, Thompson, "On the theory of visco-elasticity: a thermodynamical treatment of visco-elasticity and some problems of the vibrations of visco-elastic solids." 'Phil. Trans.,' A, vol. 231, p. 339 (1932–33).
II, "The behaviour under compression of service copper crushers," 'R.D. Report No. 64, Res. Dept. Woolwich,' H.M. Stationery Office (1927).

though superposed on, will be only of secondary magnitude as contrasted with the strains giving rise to permanent set. The two effects require types of analytical treatment having some resemblance to each other, but differing radically in that the one emphasizes the importance of compressibility and rigidity usually neglecting viscosity, the other proceeding to the limit postulates incompressibility, and stresses plasticity and viscosity. The effects will be regarded as separable analytically, and attention will be concentrated on the construction of space-time functions to deal with the first order plastico-viscous strains.

3. The arrangement observed in presentation is :—

PART I.

PHYSICAL CONSIDERATIONS AND ANALYTICAL DEVELOPMENTS.

(1) Assembly of general equations.
(2) Selection of special co-ordinates.
(3) Analogy with treatments applicable to elasticity and heat phenomena.
(4) Reduction of equation system for body motion.
(5) Boundary conditions and elastic after-recovery.
(6) Load application and formal solution.

PART II.

ANALYTICAL THEOREMS AND OBSERVATIONAL RESULTS.

(1) Comparison of analytical and experimental results (static loading).
(2) Conditions at start of motion (slow rates of compression).
(3) Comparison of analytical and experimental results (dynamic loading).
(4) Hardness and rate of compression.

PART I.

PHYSICAL CONSIDERATIONS AND ANALYTICAL DEVELOPMENTS.

1. *Assembly of General Equations.*

In assembling the material from which the functions descriptive of the motion are to be constructed benefit derives from the immediate use of the ideas and notation of the tensor calculus. A medium is to be imagined of which any particle can be located in space and time by means of three space parameters x', x^2, x^3 and the time τ, the space parameters defining a region over which the medium initially extends. Subsequent motion of the medium is then expressed in terms of functions of τ and the space parameters delimiting the region. The medium is taken to be homogeneous, isotropic, and incompressible, body forces being regarded as absent. The tensor-relationships relevant to the motion may formally be written as* :—

$$ds^2 = g_{mn}\, dx^m\, dx^n\,; \qquad (1)$$

$$p_{,t} = \frac{\partial p}{\partial x^t}\,; \qquad (2)$$

$$S^r_{s,t} = \frac{\partial S_s^r}{\partial x^t} + \left\{ {r \atop u,\, t} \right\} S_s^u - \left\{ {v \atop st} \right\} S_v^r\,; \qquad (3)$$

where

$$\left\{ {r \atop ut} \right\} = \tfrac{1}{2} g^{rw} \left\{ \frac{\partial g_{wt}}{\partial x^u} + \frac{\partial g_{wu}}{\partial x^t} - \frac{\partial g_{ut}}{\partial x^w} \right\}; \qquad (4)$$

$$e^{rs} = \tfrac{1}{2}\left\{\xi^{r,\,s} + \xi^{s,\,r}\right\}; \qquad (5)$$

$$g_{rs} e^{rs} = \xi_{,\,r}^{\;r} = 0\,; \qquad (6)$$

$$S^{tu} = -\,g^{tu} p + 2K e^{tu}\,; \qquad (7)$$

$$K = \mu + \nu \frac{d}{d\tau}\,; \qquad (8)$$

$$\nu^t = \frac{dx^t}{ds}\,. \qquad (9)$$

$$S_{,\,u}^{\;tu} = \rho f^t\,; \qquad (10)$$

$$S^r = S^{rs} \nu_s\,; \qquad (11)$$

$$f^r = \frac{d^2 \xi^r}{d\tau^2}\,; \qquad (12)$$

$$p_{,\,u,}^{\;\;u} = 0\,; \qquad (13)$$

$$K \xi_{,\,u,\,v}^{\;t,\,u,\,v} = \rho \frac{d^2}{d\tau^2}\, \xi_{,\,u}^{\;t,\,,\,u}. \qquad (14)$$

Reviewing the preceding it is to be observed that a symbol K has been introduced in equation (7) in place of the quantity conventionally designated μ. K plays an important role. It is proposed to envisage the imposition of loads of the order of 3500 kg. per cm.^{2} in times ranging upwards from the region of 10^{-3} of a second. The metal flows, the strains attain appreciable magnitudes, and viscous effects of a pronounced nature are present. Lord RAYLEIGH many years ago introduced a Dissipation

* See *e.g.*, McCONNELL, " Applications of the Absolute Differential Calculus," pp. 271 *et seq.*

Function to simulate viscous effects, and an immediate illustration may be seen by comparison of the two forms

$$\frac{d^2x}{d\tau^2} + \mu x = 0, \quad \quad (15)$$

$$\frac{d^2x}{d\tau^2} + \left(\mu + \nu \frac{d}{d\tau}\right) x = 0, \quad \quad (16)$$

equation (15) representing vibration in a normal mode, equation (16) a vibration damped through the perturbing influence of a viscous force proportional to the velocity.

Recently THOMPSON (I) has dealt with the matter in a very general way. Appealing to the principles of energy conservation, entropy, and virtual work he has demonstrated that the effects of viscosity can be satisfactorily allowed for in a scheme of equations based on normal dynamical considerations by replacing the stress constants λ, μ, by linear time operators $\lambda_1 + \lambda_2 \frac{d}{d\tau}$, $\mu_1 + \mu_2 \frac{d}{d\tau}$. By analogy the symbol K must be an operator linear in time, and its significance is therefore taken to be as expressed in equation (8). With this interpretation the stress-strain relationship (7) may be accepted as conforming to the requirements of dynamics with provision for the influence of viscosity effects. The constancy of μ and ν will be considered in Part II, and the time operator will be treated as partial. That K has now the interpretation of a linear time operator, and not a constant quantity, does not affect the established formal relationships, since K commutes with the partial derivatives with respect to the space parameters. The experimental and metallurgical aspects implying the necessity for the introduction of the time operator K are to be found in (II). They will also be touched upon in Part II of the present paper.

2. *Selection of Special Co-ordinates.*

A solid right circular cylinder of soft metal is at rest on a flat rigid surface. On the upper flat surface is a piston of rigid material. Pressure is applied to the upper end of the piston in such a way that the upper surface of the cylinder moves downwards, the soft metal deforming outwards. The motion is symmetrical. In the circumstances the natural co-ordinates for the system are cylindrical, the origin being taken at the centre of the lower flat surface, x' being the radial, x^2 the vertical, and x^3 the azimuthal parameter.

$$ds^2 = (dx')^2 + (dx^2)^2 + (x')^2 (dx^3)^2. \quad \quad (17)$$

The metric tensor is given by

$$\left.\begin{aligned} g_{11} = 1, \quad g_{22} = 1, \quad g_{33} = (x')^2, \quad g_{rs} = 0 \quad (r \neq s) \\ g^{11} = 1, \quad g^{22} = 1, \quad g^{33} = \frac{1}{(x')^2}, \quad g^{rs} = 0 \quad (r \neq s) \end{aligned}\right\} \quad \quad (18)$$

Equation (6) yields

$$\frac{\partial \xi'}{\partial x'} + \frac{\partial \xi^2}{\partial x^2} + \frac{\xi'}{x'} = 0. \quad (19)$$

Equation (7) and (5) becomes

$$-g^{ru}\frac{\partial p}{\partial x^u} + \mathrm{K}g^{st}\xi_{,st}{}^{r} = \rho\,\frac{d^2\xi^r}{d\tau^2}. \quad (20)$$

Of the three equations under (20) one disappears, from symmetry, and two survive. They are

$$-\frac{\partial p}{\partial x'} + \mathrm{K}\left\{\frac{\partial^2\xi'}{\partial (x')^2} + \frac{\partial^2\xi'}{\partial (x^2)^2} - \frac{\xi'}{(x')^2} + \frac{1}{x'}\cdot\frac{\partial \xi'}{\partial x'}\right\} = \rho\,\frac{d^2\xi'}{d\tau^2}, \quad (21)$$

$$-\frac{\partial p}{\partial x^2} + \mathrm{K}\left\{\frac{\partial^2\xi^2}{\partial (x')^2} + \frac{\partial^2\xi^2}{\partial (x^2)^2} + \frac{1}{x'}\cdot\frac{\partial \xi^2}{\partial x'}\right\} = \rho\,\frac{d^2\xi^2}{d\tau^2}. \quad (22)$$

Equation (13) indicates that one of the above equations may be replaced by

$$\frac{\partial^2 p}{\partial (x')^2} + \frac{\partial^2 p}{\partial (x^2)^2} + \frac{1}{x'}\cdot\frac{\partial p}{\partial x'} = 0. \quad (23)$$

Equations (22) and (23) as the least involved are taken to be the two equations representative of general body motion.

In motions of the type now under consideration the equations of surface traction are of an importance equal to that of the equations of body motion. The connection between a surface traction S^r and a body stress is given in equation (11).

3. *Analogy with Treatments applicable to Elasticity and Heat Phenomena.*

Before proceeding to the construction of functions to satisfy the equations of body motion and the boundary conditions it is of help to consider physical aspects. They are two-fold and may perhaps best be approached by generalizing from the treatment normally accorded to elasticity problems and also from the procedure employed when appeal is made to the principle of entropy. In elasticity problems a strained body is assumed to be capable of returning by the reverse path to an initially unstrained state, which is taken as standard. The strains are small, and when they exceed a specific limit, varying for different media, the body acquires a permanent set, and is said to be strained beyond the elastic limit. In general, in such circumstances, its resistance to further deformation is reduced. Now the soft metal at present under discussion definitely exhibits elastic properties provided the strains are sufficiently small. The intention, however, is to proceed beyond the small strain limits immediately, and to consider the position from and after the acquiry of initial permanent set. For the starting conditions it is assumed that the medium has already acquired some permanent set, and that it may also be in motion in some prescribed manner. Not all conceivable motions are possible, but only such motions as have been attained following some prescribed path. Although reference is made to starting conditions as standard there can be no return to them, the motion is strictly irreversible. On the other hand, there are compensations in that the permanent set is retained and in great measure ascertainable, as also possible changes in other physical characteristics, for example hardness. The notion of irreversibility of path leads naturally to the consideration of the second physical aspect. A difference phenomenon is under review, and the cylinder of soft metal may be regarded as a working body to which is applied pressure resulting in change of shape and liberation of heat, which, for the short times to be considered, may be presumed to be retained in the body. There is a store of energy latent, and this the body re-allocates to compete with the applied stresses. THOMPSON's treatment (I) has shown how entropy changes may be allowed for by the introduction of the linear time operator K; and that, with the modification, the equations may be accepted as possessing the usual dynamical significance. The immediate concern is a molar difference phenomenon, the real basis is inter-molecular readjustment.

4. *Reduction of Equation System for Body Motion.*

Recasting the equations into a more helpful form, let the initial radius of the cylinder be R, and the initial height H. Write

$$x' = \mathrm{R}r, \qquad x^2 = \mathrm{H}h$$

$$\xi' = \mathrm{RF}^r, \quad \xi^2 = \mathrm{HF}^h.$$

The quantities r, h, F^r, F^h are pure number variables.
Equation (23) becomes

$$\frac{\partial^2 p}{\partial h^2} + \frac{1}{b^2}\left(\frac{\partial^2 p}{\partial r^2} + \frac{1}{r}\cdot\frac{\partial p}{\partial r}\right) = 0 \qquad \left(b = \frac{\mathrm{R}}{\mathrm{H}}\right). \quad \ldots \ldots \quad (24)$$

Equation (22) becomes,

$$-\frac{\partial p}{\partial h} + \mathrm{K}\left\{\frac{\partial^2 \mathrm{F}^h}{\partial h^2} + \frac{1}{b^2}\left(\frac{\partial^2 \mathrm{F}^h}{\partial r^2} + \frac{1}{r}\,\frac{\partial \mathrm{F}^h}{\partial r}\right)\right\} = \rho \mathrm{H}^2 \frac{d^2 \mathrm{F}^h}{d\tau^2}. \quad \ldots \ldots \quad (25)$$

Equation (19)

$$\frac{\partial \mathrm{F}^h}{\partial h} + \frac{\partial \mathrm{F}^r}{\partial r} + \frac{\mathrm{F}^r}{r} = 0. \quad \ldots \ldots \ldots \ldots \quad (26)$$

The presence of the operator K and the fact that freedom must be reserved to impose any type of loading condition suggests avoidance of any special types of functions in an attempted solution. The functions to be employed are therefore considered to have a general form, and to be capable of being expressed as double series in powers of r and h, with time functions as coefficients. It will be observed that a solution will be reached by a procedure somewhat the inverse of that pertaining to normal dynamical problems. In Dynamics, being in possession of the equations of motion, it is generally possible to proceed from the starting conditions and the leading equations to the solution of subsidiary equations. In the present procedure it is found necessary first to examine the subsidiary equations, move thence upwards to the leading equations, introduce the important boundary and loading conditions, and arrive finally at the integrating functions and constants after a complete survey of the whole circumstances of the motion. It is not possible at the start to lay down any conditions that certain functions may be equated to zero. It is necessary to examine whether the equations of motion can be satisfied if such conditions be imposed. The route is circuitous but unavoidable.

Now equation (24) may be satisfied by writing

$$\begin{aligned} p = & \left\{\alpha + \frac{\beta}{\lfloor 1}h + \frac{\gamma}{\lfloor 2}h^2 + \frac{\delta}{\lfloor 3}h^3 + \frac{\varepsilon}{\lfloor 4}h^4 + \dots\right\} \\ & - \frac{b^2}{2^2}r^2\left\{\gamma + \frac{\delta}{\lfloor 1}h + \frac{\varepsilon}{\lfloor 2}h^2 + \dots\right\} \\ & + \frac{b^4}{2^2 \cdot 4^2}r^4\left\{\varepsilon + \frac{\zeta}{\lfloor 1}h + \dots\right\} \\ & - \frac{b^6}{2^2 \cdot 4^2 \cdot 6^2}r^6\left\{\eta + \dots\right\} \\ & + \dots, \qquad (27) \end{aligned}$$

where the α, β, ..., are time functions.

In keeping with the form of (27) write

$$-\mathrm{F}^h = \mathrm{F}_0(h) + \frac{b^{2s}}{2^2 \cdot 4^2 \dots (2s)^2}(r)^{2s}\mathrm{F}_s(h), \qquad (28)$$

where the second term symbolizes a summation of terms from $s = 1$ upwards. $\mathrm{F}_s(h)$ (all s) are functions in ascending powers of h with time functions as coefficients. The negative sign is attached to F^h since it is essentially a negative quantity; and there is need, as far as feasible, to keep the $\mathrm{F}_s(h)$ functions positive.

$$-\frac{\partial \mathrm{F}^h}{\partial h} = \mathrm{F}'_0(h) + \frac{b^{2s}}{2^2 \cdot 4^2 \dots (2s)^2}(r)^{2s}\,\mathrm{F}'_s(h), \qquad (29)$$

$$-\frac{\partial^2 \mathrm{F}^h}{\partial h^2} = \mathrm{F}''_0(h) + \frac{b^{2s}}{2^2 \cdot 4^2 \dots (2s)^2}(r)^{2s}\,\mathrm{F}''_s(h), \qquad (30)$$

$$-\frac{\partial \mathrm{F}^h}{\partial r} = 2s \cdot \frac{b^{2s}}{2^2 \cdot 4^2 \dots (2s)^2}(r)^{2s-1}\,\mathrm{F}_s(h), \qquad (31)$$

$$-\frac{\partial^2 \mathrm{F}^h}{\partial r^2} = 2s\,(2s-1)\frac{b^{2s}}{2^2 \cdot 4^2 \dots (2s)^2}(r)^{2s-2}\,\mathrm{F}_s(h). \qquad (32)$$

From equation (26)

$$-\frac{\partial F^h}{\partial h} = \frac{1}{r} \cdot \frac{\partial}{\partial r}(rF^r). \qquad (33)$$

Therefore from (29)

$$\left.\begin{aligned} \frac{\partial}{\partial r}(rF^r) &= rF'_0(h) + \frac{b^{2s}}{2^2 \cdot 4^2 \ldots (2s)^2}(r)^{2s+1} F'_s(h) \\ \text{or} \qquad rF^r &= \frac{r^2}{2} F'_0(h) + \frac{1}{2s+2} \frac{b^{2s}}{2^2 \cdot 4^2 \ldots (2s)^2}(r)^{2s+2} \cdot F'_s(h) + \chi(h) \end{aligned}\right\}, \qquad (34)$$

where $\chi(h)$ is a function of integration, therefore

$$F^r = \frac{r}{2} F'_0(h) + \frac{1}{2s+2} \cdot \frac{b^{2s}}{2^2 \cdot 4^2 \ldots (2s)^2}(r)^{2s+1} F'_s(h), \qquad (35)$$

the function of integration disappearing since F^r is zero for $r = 0$ (all h).

Therefore

$$\frac{\partial F^r}{\partial h} = \frac{r}{2} F''_0(h) + \frac{1}{2s+2} \frac{b^{2s}}{2^2 \cdot 4^2 \ldots (2s)^2}(r)^{2s+1} F''_s(h) \qquad (36)$$

$$\frac{\partial F^r}{\partial r} = \tfrac{1}{2} F'_0(h) + \frac{2s+1}{2s+2} \cdot \frac{b^{2s}}{2^2 \cdot 4^2 \ldots (2s)^2}(r)^{2s} \cdot F'_s(h). \qquad (37)$$

The function $F_s(h)$ is defined (all s) to be

$$F_s(h) = F_{st} \cdot \frac{h^t}{\underline{|t}}, \qquad (38)$$

the term on the right implying a summation of terms from $t = 1$, the F_{st} being time functions.

Substituting from the above equations in (25) and equating series coefficients of corresponding powers of r there results

$$\left\{\beta + \frac{\gamma}{\underline{|1}} h + \frac{\delta}{\underline{|2}} h^2 + \ldots\right\} + K\{F''_0(h) + F_1(h)\} = \rho H^2 D^2 F_0(h) \qquad (39)$$

$$-\left\{\delta + \frac{\varepsilon}{\underline{|1}} h + \ldots\right\} + K\{F''_1(h) + F_2(h)\} = \rho H^2 D^2 F_1(h) \qquad (40)$$

$$\{\zeta + \ldots\} + K\{F''_2(h) + F_3(h)\} = \rho H^2 D^2 F_2(h) \qquad (41)$$

$$\ldots\ldots = \ldots,$$

where D has been written for the time operator $d/d\tau$.

Equating coefficients of corresponding powers of h in (39) onwards there follows

$$\left.\begin{aligned} \beta + \mathrm{KF}_{02} &= 0 \\ \frac{\gamma}{\lfloor 1} + \mathrm{K}\left[\frac{\mathrm{F}_{03}}{\lfloor 1} + \frac{\mathrm{F}_{11}}{\lfloor 1}\right] &= \rho \mathrm{H}^2\mathrm{D}^2 \frac{\mathrm{F}_{01}}{\lfloor 1} \\ \frac{\delta}{\lfloor 2} + \mathrm{K}\left[\frac{\mathrm{F}_{04}}{\lfloor 2} + \frac{\mathrm{F}_{12}}{\lfloor 2}\right] &= \rho \mathrm{H}^2\mathrm{D}^2 \frac{\mathrm{F}_{02}}{\lfloor 2} \\ \ldots\ldots &= \ldots \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (42)$$

$$\left.\begin{aligned} -\delta + \mathrm{KF}_{12} &= 0 \\ -\frac{\varepsilon}{\lfloor 1} + \mathrm{K}\left[\frac{\mathrm{F}_{13}}{\lfloor 1} + \frac{\mathrm{F}_{21}}{\lfloor 1}\right] &= \rho \mathrm{H}^2\mathrm{D}^2 \frac{\mathrm{F}_{11}}{\lfloor 1} \\ -\frac{\zeta}{\lfloor 2} + \mathrm{K}\left[\frac{\mathrm{F}_{14}}{\lfloor 2} + \frac{\mathrm{F}_{22}}{\lfloor 2}\right] &= \rho \mathrm{H}^2\mathrm{D}^2 \frac{\mathrm{F}_{12}}{\lfloor 2} \\ \ldots\ldots &= \ldots \end{aligned}\right\} \quad \ldots\ldots\ldots \quad (43)$$

$$\left.\begin{aligned} \zeta + \mathrm{KF}_{22} &= 0 \\ \eta + \mathrm{K}\left[\frac{\mathrm{F}_{23}}{\lfloor 1} + \frac{\mathrm{F}_{31}}{\lfloor 1}\right] &= \rho \mathrm{H}^2\mathrm{D}^2\mathrm{F}_{21} \\ \ldots\ldots &= \ldots \end{aligned}\right\} \ldots\ldots\ldots \quad (44)$$

The inverse nature of the unavoidable procedure is now becoming more apparent. To admit freedom of choice to impose on the starting conditions any permissible displacement and velocity the F_{st} functions, when integrated, must contain two free constants. For these to be available it is necessary to integrate equations of the second order in time, but as presented, any second order equation for an F_{st} depends upon subsequent, not previous, functions in the series, and in addition the values of the time functions typified by the Greek letters also require to be known in an explicit form. To make progress there is appeal to the boundary conditions, which may be concerned either with geometry, or stress, or both. They are important, and exercise a marked effect on the motion. In addition to the loading, which is at choice, there is also the possibility, by circular scribing or otherwise, of imposing on the flat surface radial displacement any magnitude from complete restriction to complete freedom. It is hoped to deal with the first type of constraint on some future occasion, noting meanwhile that the restriction is imposed by requiring

$$\mathrm{F}'_t(h) = 0 \qquad (\text{all } t,\ h = 0, 1).$$

Barrelling will ensue from the start of compression consequent on the incompressibility of the medium, and the motion becomes complex when the tangent planes at the upper and lower edges of the curved boundary become horizontal. Intermediate types of friction grip are as difficult to specify analytically as they are conducive to unsatisfactory results in practice. It is proposed to examine the conditions briefly, in due course, and then to limit consideration to the most useful mode of compression, namely, with surface friction effectively eliminated. No tangential traction on the upper and lower flat surfaces imposes the requirement

$$K\left(\frac{\partial \xi'}{\partial x^2}+\frac{\partial \xi^2}{\partial x'}\right)=0 \qquad (h=0, 1\;;\ \text{all } r). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (45)$$

Further, the geometrical constraint that the upper and lower surfaces remain plane during the compression entails

$$F_t(h)=0 \qquad (h=0, 1\;;\ \text{all } t>0). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (46)$$

Turning now to the curved boundary the physical condition is that there is no applied external load; in other words, the matter at the periphery is free to move sideways so far as internal surface tension effects will permit. Formally considered the constraint should be that all curved boundary stresses vanish. There are three conditions to satisfy arising from equation (11). It will appear subsequently that for the main problem considered two of the conditions can be satisfied immediately; but that the third raises the question of elastic after-recovery. There is also another interesting phenomenon to be noted at the curved boundary. For small compressions the periphery remains smooth; as the compressions are increased the surface becomes rougher, assuming finally for large compressions a crinkled and nodular appearance. Of this the analysis, as developed, can take no account.

5. *Boundary Conditions and Elastic After-recovery.*

In endeavouring to reach a solution satisfying all the requirements noted previously detailed consideration of equations (45) and (46) brings to light that the terms in the power series for F_s (all s) exhibit characteristically different properties; the odd powers of h are terms having reference more specifically to the applied load, the even powers to tangential surface tractions. Nothing can be done with the equation systems (42), (43), (44), until the explicit forms of the time functions α, β, γ, . . ., are known, or equations are available for their determination. The α, γ, ε, . . ., functions can be ascertained through the curved boundary and loading conditions. The β, δ, ζ, . . ., functions are required to specify the nature of the tangential surface tractions. There would appear to be no means available for accurate specification of intermediate friction conditions and the quantities accordingly will all be equated to zero, thus restricting consideration to the main problem of compression with no flat-surface friction, the condition normally aimed at in practice. With the vanishing of β, δ, ζ, . . ., all the F_{tu} time functions (u even) vanish from the equation systems (42), (43), (44) since the integrated time functions, if existent, must contain two free constants of integration.

There remain consequently the α, γ, ε, . . . , time functions with the F_{st} functions correlated to them (t odd).

Now equation (45) written at length implies

$$\left.\begin{aligned} KF''_0(1) - KF_1(1) &= 0 \\ KF''_1(1) - KF_2(1) &= 0 \\ \ldots\ldots &= \ldots \end{aligned}\right\} \qquad (47)$$

Equation (46) requires that

$$F_1(1) = F_2(1) = \ldots = 0. \qquad (48)$$

Recalling that the above equations obtain throughout the motion, and that K is a linear time operator, equations (48) entail

$$KF_1(1) = KF_2(1) = \ldots = 0. \qquad (49)$$

Consequently in addition to (48) using (49) in (47)

$$KF''_0(1) = KF''_1(1) = \ldots = 0. \qquad (50)$$

At $h = 0$ conditions are immediately satisfied, since all the function series in question have h as the lowest retained power.

Consider now the equation systems similar to (43), (44) onwards in detail and add the equations vertically. From (49) and (50)

$$\left.\begin{aligned} \frac{\varepsilon}{\lfloor 1} + \frac{\eta}{\lfloor 3} + \frac{\iota}{\lfloor 5} + \ldots &= 0 \\ \frac{\eta}{\lfloor 1} + \frac{\iota}{\lfloor 3} + \ldots &= 0 \\ \frac{\iota}{\lfloor 1} + \ldots &= 0 \\ \ldots\ldots &= \ldots \end{aligned}\right\} \qquad (51)$$

Rejecting the possibility of satisfying the equations by resort to powers of π and some general time function the solution of (51) is taken to be

$$\varepsilon = \eta = \iota = \ldots = 0. \qquad (52)$$

With the vanishing of ε, η, ι, . . ., the equation systems (43), (44) can be satisfied by writing

$$F_{rs} = 0 \qquad (\text{all } s\,; \ \text{all } r > 0)$$

since no function can be integrated in a manner to retain two free constants of integration.

Outstanding is the reduced equation system (42), namely,

$$\left.\begin{aligned} \frac{\gamma}{\lfloor 1} + \mathrm{K}\frac{\mathrm{F}_{03}}{\lfloor 1} &= \rho \mathrm{H}^2\mathrm{D}^2\frac{\mathrm{F}_{01}}{\lfloor 1} \\ \mathrm{K}\frac{\mathrm{F}_{05}}{\lfloor 3} &= \rho \mathrm{H}^2\mathrm{D}^2\frac{\mathrm{F}_{03}}{\lfloor 3} \\ \ldots &= \ldots \end{aligned}\right\} \qquad (53)$$

Adding the system of equations vertically there results

$$\gamma = \rho \mathrm{H}^2\mathrm{D}^2\left\{\frac{\mathrm{F}_{01}}{\lfloor 1} + \frac{\mathrm{F}_{03}}{\lfloor 3} + \ldots\right\}, \qquad (54)$$

since the sum of the KF_{03} column vanishes from equation (50).

A solution to satisfy (53) and (54) is reached by taking

$$\begin{aligned} \gamma &= \rho \mathrm{H}^2\mathrm{D}^2\mathrm{F}_{01} \\ \mathrm{F}_{03} = \mathrm{F}_{05} &= \ldots = 0. \end{aligned} \qquad (55)$$

Reviewing the development so far it is seen that satisfaction of the tangential traction and flat surface conditions at the ends has placed serious restrictions on the equations of body motion. Only three time functions are available, namely, α, γ, and F_{01}, and between two of these a relation subsists. Only two more equations can be used to ascertain the quantities; and there are three conditions outstanding on the curved boundary, and the loading condition on the upper flat surface.

From equation (11) the stress at the curved surface has three components which, from symmetry, may be written

$$\mathrm{S}^{11}l + \mathrm{S}^{12}m, \quad \mathrm{S}^{21}l + \mathrm{S}^{22}m, \quad \mathrm{S}^{31}l + \mathrm{S}^{32}m, \qquad (56)$$

where l, m, are direction cosines in an axial plane. From (55) since F_{01} is the one time function outstanding to measure strain the cylinder remains cylindrical, consequently $m = 0$. The function S^{31} vanishes, and $\mathrm{S}^{21} = \mathrm{S}^{12} =$ zero (from 45). Two of the components vanish by reason of conditions already imposed and one remains—S^{11}— since $l = 1$. If S^{11} $(r = 1)$ be equated to zero there results

$$-\left(\alpha + \frac{\gamma h^2}{\lfloor 2} - \frac{b^2}{2^2}\gamma\right) + \mathrm{KF}_{01} = 0. \qquad (57)$$

Three of the terms are time functions and the fourth is the product of a time function and a space variable. To satisfy the equation formally therefore there should be written

$$\alpha = \mathrm{KF}_{01} + \frac{b^2}{2^2}\gamma, \qquad (58)$$

$$\gamma = 0\,; \quad \text{or} \quad h = 0. \qquad (59)$$

Now in (59) the condition $h = 0$ ($r = 1$) holds only at the lower rim of the periphery; further from (55) the condition $\gamma = 0$ implies rest or no acceleration. Thus strictly interpreted equation (58) can only be accepted as holding in general at the lower rim, and as holding in particular over the whole curved surface when there is rest or no acceleration. It would seem that the term $\left(-\gamma \frac{h^2}{\lfloor 2}\right)$ is the measure of an outstanding stress at the periphery. During acceleration from (55) γ is positive and the outstanding stress is a pressure, during retardation γ is negative and the outstanding stress is a traction. Qualitatively considered they would be precisely the types of stress required to deal with a fictitious layer adhering to the surface to retain it in position during motion.

It is now necessary to enquire the order of error likely to be introduced by neglect of the term including the space variable. Rewrite (57) as

$$\alpha - \frac{b^2}{2^2}\gamma = \mathrm{KF}_{01}\left\{1 - \tfrac{1}{2}\frac{\gamma h^2}{\mathrm{KF}_{01}}\right\},$$

where the greatest value of h is unity, and

$$\gamma = \rho \mathrm{H}^2\mathrm{D}^2\mathrm{F}_{01}. \quad \ldots \ldots \ldots \ldots (55)$$

The quantity under investigation is of the order

$$\tfrac{1}{2}\frac{\rho\, \mathrm{H}^2\mathrm{D}^2\mathrm{F}_{01}}{\mathrm{KF}_{01}}.$$

In Part II it will appear that, under plastic conditions, K has a value in the region of 2×10^9 C.G.S. units. If F_{01} be simulated by a $\sin n\tau$ term, where the time to maximum loading is given by $n\tau = \frac{1}{2}\pi$,

$$\tfrac{1}{2}\frac{\rho\, \mathrm{H}^2\mathrm{D}^2\mathrm{F}_{01}}{\mathrm{KF}_{01}}$$

is of the order

$$\tfrac{1}{2}\cdot\frac{10n^2}{2\times 10^9} \qquad (\mathrm{H} = 1 \text{ cm.}).$$

For a value of $n^2 \ngtr 10^7$ the neglect of such a term will entail no error greater than is comprised in the initial neglect of squares of first order quantities. Times to maximum loading even so low as 5×10^{-4} seconds may consequently be contemplated without serious fear that the acceptance of equation (58) as holding not solely at the lower rim but over the whole curved boundary will not be in order within the prescribed limits of accuracy.

A third aspect remains for consideration. The solution has now been sufficiently developed to deal with a cylinder at rest under load. It is assumed that there has been appreciable strain, and that the position has become balanced. From the curved boundary condition

$$\alpha = \mathrm{KF}_{01}. \quad \ldots \ldots \ldots \ldots (60)$$

From the upper flat surface condition, R being the radius, and L the loading

$$\{\alpha + 2\mathrm{KF}_{01}\}\,\pi\mathrm{R}^2 = \mathrm{L} \quad \ldots\ldots\ldots\ldots \quad (61)$$

K now takes the form μ since the cylinder is at rest. Therefore

$$\mathrm{F}_{01} = \frac{\mathrm{L}}{3\mu\pi.\mathrm{R}^2} = \frac{x^2}{\mathrm{H}}. \quad \ldots\ldots\ldots\ldots \quad (62)$$

The quantity x^2 measures the downward compression.

Remove the load, and re-examine the conditions

$$\alpha' = \mathrm{KF}_{01} \quad \ldots\ldots\ldots\ldots \quad (60')$$

$$\alpha' + 2\mathrm{KF}_{01} = 0. \quad \ldots\ldots\ldots\ldots \quad (61')$$

Now F_{01} cannot become zero; that would imply the vanishing of the plastic strain which does not happen. The removal of the load entails the disappearance of α, and at the curved surface there is an outstanding tension KF_{01}, whilst at the upper flat surface there is an outstanding pressure $(-2\mathrm{KF}_{01})$. It is necessary to appeal to the known phenomenon of elastic after-recovery to reconcile the position. The material at the curved boundary moves inwards to a slight extent whilst the material at the higher flat surface moves upwards, thus relieving the plastic stresses. In so doing there is brought into play elastic strain to compensate. It is to be observed that the plasticity coefficient is of the order 2×10^9, whilst the rigidity coefficient for elastic strain is known to be of the order 4×10^{11} C.G.S. units, thus for the same strain magnitude the stresses at issue are of the order 2×10^2 elastically : 1 plastically. An elastic after-recovery of the order of $0{\cdot}5\%$ of the plastic deformation will thus bring into play balancing stresses and the medium will be at rest under the superposed plastic and elastic recovery strains. The phenomenon is of importance not only at the beginning and end of plastic deformation, but also when load is reapplied to a previously compressed cylinder, the operation known as "check-pointing" (II). On reapplication of load the first motion is gradual elastic compression until the stage is reached indicated under equation (62). On increase of load additional permanent set ensues. By analogy it would seem that a phenomenon of this nature is in action at the free curved boundary during motion; and that if the idealized plastic requirement—at free boundaries surface-traction vanishes—be replaced by the less restrictive, but substantially equivalent postulate—at free boundaries during both rest and motion stress vanishes in the sense that residual plastic stress brings into action compensating elastic strain—conditions can be formally satisfied, and at the same time the linkage phenomenon of elastic recovery not be overlooked. To a postulate of this kind we appear to be unavoidably led by reason that elastic and plastic conditions though co-existing cannot analytically be treated concurrently, where the former requires compressibility, and the latter assumes incompressibility of the medium. Having regard to all the considerations it would appear that

$$\alpha = \mathrm{KF}_{01} + \frac{b^2}{2^2}\gamma \quad \ldots \ldots \ldots \ldots \quad (58)$$

may be accepted for plastic strain purposes as the equation furnished by conditions at the curved boundary.

The inference from (55) is that barrelling will not occur if there be no friction at the flat surfaces, and experimental evidence may be adduced in support. Provided the flat surfaces are rendered sufficiently smooth barrelling does not occur in practice until the cylinder has been compressed down by at least one-fourth of its initial height. Barrelling then begins to be manifest, and progressively increases as the compression increases. Since increased compression is necessarily associated with increased pressure it is presumable that the friction, which in the conditions may to some extent be proportional to the pressure, is with sustained compression, attaining to a magnitude sufficient to act appreciably as a tangential surface traction to the flat surfaces.

Assembling the conditions outstanding so far :—

$$\gamma = \rho \mathrm{H}^2\mathrm{D}^2\mathrm{F}_{01} \quad \ldots \ldots \ldots \ldots \quad (55)$$

$$\alpha = \mathrm{KF}_{01} + \frac{b^2}{2^2}\gamma. \quad \ldots \ldots \ldots \ldots \quad (58)$$

All F_{rs} other than F_{01} vanish; the motion is cylindrical: planes remain planes; tangential surface tractions vanish: stress vanishes on the curved boundary.

6. *Load Application and Formal Solution.*

The important matter of load application may now be considered. Investigation of the plastic motion attendant on weight dropping must be deferred to some future occasion, but the phenomenon of dynamic overshoot can be taken into the equation scheme without adding materially to the complexity. Resting on the upper surface of the soft metal cylinder is a piston of mass M, with a plane face. The load is applied to the upper plane surface of the piston.

Now the traction normal to the surface at any point in the upper flat of the soft metal cylinder is given by

$$-p + 2\mathrm{K}\frac{\partial \xi^2}{\partial x^2} \qquad (h = 1),$$

that is, by

$$-\left\{\alpha + \frac{\gamma}{\lfloor 2} - \frac{b^2}{2^2}r^2\gamma\right\} - 2\mathrm{KF}_{01}. \quad \ldots \ldots \ldots \ldots \quad (63)$$

This is the force acting upwards on the flat surface of the cylinder. It is therefore the force acting downwards at the corresponding point of the lower piston flat. Action is at the point $(r, 1)$, and the integrated expression needs to be used for the total force applicable to the piston.

Expression (63) integrated from $r = 0$ to $r = 1$ is :—

$$-\left[\left\{\alpha + \frac{\gamma}{\underline{|2}} + 2\mathrm{KF}_{01}\right\} - \tfrac{1}{2}\cdot\frac{b^2}{2^2}\,\gamma\right]\pi\mathrm{R}^2\,(1 + \mathrm{F}_{01}),$$

since

$$2\pi\,(x' + \xi')\,d\,(x' + \xi') = 2\pi\mathrm{R}^2\,(1 + \mathrm{F}_{01})\,rdr \quad \text{(neglecting squares } h = 1).$$

Let $\mathrm{L}(\tau)$ be the load on the upper surface of the piston, then

$$\mathrm{L}\,(\tau) - \left[\left\{\alpha + \frac{\gamma}{\underline{|2}} + 2\mathrm{KF}_{01}\right\} - \tfrac{1}{2}\cdot\frac{b^2}{2^2}\,\gamma\right]\pi\mathrm{R}^2\,(1 + \mathrm{F}_{01}) = -\,\mathrm{MD}^2\xi^2. \quad . \quad (64)$$

Now

$$\xi^2 = \mathrm{HF}^h = -\,\mathrm{HF}_{01}, \qquad (h = 1),$$

therefore using (55) and (58) :

$$3\mathrm{KF}_{01} + \left\{\frac{\mathrm{M}}{\pi\mathrm{R}^2\mathrm{H}\rho} + \tfrac{1}{2}\left(1 + \frac{b^2}{2^2}\right)\right\}\rho\mathrm{H}^2\mathrm{D}^2\mathrm{F}_{01} = \frac{\mathrm{L}\,(\tau)}{\pi\mathrm{R}^2}\{1 - \mathrm{F}_{01}\}. \quad . \quad . \quad . \quad (65)$$

For brevity write

$$\mathrm{N} = \frac{\mathrm{M}}{\pi\mathrm{R}^2\mathrm{H}\rho} + \tfrac{1}{2}\left(1 + \frac{b^2}{2^2}\right), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (66)$$

the significance of N is interesting. It contains two terms the first being the ratio of the mass of the piston to the mass of the cylinder, the second combined with it additively contains the shape constant $b = \mathrm{R}/\mathrm{H}$. N is a pure number and increases or decreases as M increases or decreases compared with the mass of the cylinder, and as R increases or decreases compared with H. Experimentally it is found advantageous to keep both M and b as small as possible.

A first approximation to the solution of (65) is given by

$$\mathrm{F}_{01} = \mathrm{I}_1 e^{-\alpha_1\tau} + \mathrm{I}_2 e^{-\alpha_2\tau} + \frac{1}{3\mathrm{K} + \mathrm{N}\rho\mathrm{H}^2\mathrm{D}^2}\cdot\frac{\mathrm{L}\,(\tau)}{\pi\mathrm{R}^2}, \quad . \quad . \quad . \quad . \quad . \quad (67)$$

where α_1, α_2 are the roots of

$$\frac{\mathrm{N}}{3}\,\rho\mathrm{H}^2\alpha^2 - \alpha\nu + \mu = 0. \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (68)$$

Of the roots, α_1 is taken to be the greater, and consequently

$$\left.\begin{aligned}\alpha_1 &= \frac{3\nu}{2\mathrm{N}\rho\mathrm{H}^2}\left\{1 + \left(1 - \frac{4\mathrm{N}\rho\mathrm{H}^2\mu}{3\nu^2}\right)^{\frac{1}{2}}\right\}\\ \alpha_2 &= \frac{3\nu}{2\mathrm{N}\rho\mathrm{H}^2}\left\{1 - \left(1 - \frac{4\mathrm{N}\rho\mathrm{H}^2\mu}{3\nu^2}\right)^{\frac{1}{2}}\right\}\end{aligned}\right\} \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (69)$$

Attention so far, experimentally, has never been strongly concentrated on the RAYLEIGH-THOMPSON viscous coefficient ν, but such evidence as exists indicates that its magnitude is of the order 2×10^6 C.G.S. units. The magnitude of μ under plastic conditions is of the order 2×10^9 C.G.S. units. Unless therefore N is exceptionally high the expressions, as written under (69), are real and positive

$$\alpha_1 \rightarrow \frac{3\nu}{N\rho H^2}, \qquad \alpha_2 \rightarrow \frac{\mu}{\nu}. \quad \ldots \quad (70)$$

The condition—N very large—is so far removed from practice that the possibility of occurrence of heavily damped oscillations will not be investigated.

Proceeding to a second approximation to the solution of (65) write the right-hand side of equation (67) in the factor attaching to $L(\tau)/\pi R^2$ in the right-hand side of equation (65) and integrate using the same general integral.

Consequently

$$F_{01} = I_1 e^{-\alpha_1\tau} + I_2 e^{-\alpha_2\tau} + \frac{1}{3K + N\rho H^2 D^2} \cdot \frac{L(\tau)}{\pi R^2}$$
$$- \frac{1}{3K + N\rho H^2 D^2} \left[\frac{L(\tau)}{\pi R^2} \left\{ I_1 e^{-\alpha_1\tau} + I_2 e^{-\alpha_2\tau} + \frac{1}{3K + N\rho H^2 D^2} \cdot \frac{L(\tau)}{\pi R^2} \right\}\right], \quad (71)$$

Using the transformation

$$F(D)\, e^{a\tau} T = e^{a\tau} F(D + a)\, T,$$

where a is a constant and T is a function of τ, a simplification may be introduced as follows :—

$$\frac{1}{3K + N\rho H^2 D^2} e^{-\alpha_1\tau} . L(\tau) = e^{-\alpha_1\tau} \frac{1}{3[\mu + \nu(D - \alpha_1)] + N\rho H^2 (D - \alpha_1)^2} . L(\tau)$$
$$= e^{-\alpha_1\tau} \frac{1}{3(\mu - \nu\alpha) + N\rho H^2 \alpha_1^2 + (3\nu - 2N\rho H^2 \alpha_1) D + N\rho H^2 D^2} L(\tau)$$
$$= e^{-\alpha_1\tau} \frac{1}{(3\nu - 2N\rho H^2 \alpha_1) D + N\rho H^2 D^2} . L(\tau)$$

(using (68)).

Therefore

$$F_{01} = I_1 e^{-\alpha_1\tau} + I_2 e^{-\alpha_2\tau} + \frac{1}{3K + N\rho H^2 D^2} \cdot \frac{L(\tau)}{\pi R^2}$$
$$- I_1 e^{-\alpha_1\tau} \frac{1}{(3\nu - 2N\rho H^2 \alpha_1) D + N\rho H^2 D^2} \cdot \frac{L(\tau)}{\pi R^2}$$
$$- I_2 e^{-\alpha_2\tau} \frac{1}{(3\nu - 2N\rho H^2 \alpha_2) D + N\rho H^2 D^2} \cdot \frac{L(\tau)}{\pi R^2}$$
$$- \frac{1}{3K + N\rho H^2 D^2} \cdot \left[\frac{L(\tau)}{\pi R^2} \cdot \frac{1}{3K + N\rho H^2 D^2} \cdot \frac{L(\tau)}{\pi R^2}\right] \quad \ldots \quad (72)$$

The above equation completes the search for a formal solution. The load function L (τ) is at choice, and the integrating constants I_1, I_2, admit of specific values being accorded to F_{01} and DF_{01} at a particular instant.

With primitive forms of L (τ) no difficulty is likely to be experienced in arriving at explicit expressions for F_{01} and DF_{01}. More complicated L (τ) functions can be expressed in appropriate Fourier Series, and as a typical term in such a series, also for subsequent purposes, there may be taken

$$L(\tau) = L_0 \sin n\tau, \qquad (73)$$

where L_0 is a fixed quantity, and maximum loading conditions are reached after a time interval $\pi/2n$ seconds from the start of loading. Omitting transformations

$$\begin{aligned} F_{01} = {} & I_1 e^{-\alpha_1\tau} + I_2 e^{-\alpha_2\tau} + \frac{(3\mu - N\rho H^2 n^2)\sin n\tau - 3\nu n \cos n\tau}{(3\mu - N\rho H^2 n^2)^2 + (3\nu n)^2} \cdot \frac{L_0}{\pi R^2} \\ & + \left[\frac{n(3\nu - 2N\rho H^2\alpha_1) I_1 e^{-\alpha_1\tau}}{n^2(3\nu - 2N\rho H^2\alpha_1)^2 + (N\rho H^2 n^2)^2} + \frac{n(3\nu - 2N\rho H^2\alpha_2) I_2 e^{-\alpha_2\tau}}{n^2(3\nu - 2N\rho H^2\alpha_2)^2 + (N\rho H^2 n^2)^2}\right] \cos n\tau \cdot \frac{L_0}{\pi R^2} \\ & + \left[\frac{N\rho H^2 n^2 I_1 e^{-\alpha_1\tau}}{n^2(3\nu - 2N\rho H^2\alpha_1)^2 + (N\rho H^2 n^2)^2} + \frac{N\rho H^2 n^2 I_2 e^{-\alpha_2\tau}}{n^2(3\nu - 2N\rho H^2\alpha_2)^2 + (N\rho H^2 n^2)^2}\right] \sin n\tau \cdot \frac{L_0}{\pi R^2} \\ & - \frac{(3\mu - N\rho H^2 n^2)}{6\mu[(3\mu - N\rho H^2 n^2)^2 + (3\nu n)^2]} \cdot \left(\frac{L_0}{\pi R^2}\right)^2 \\ & + \frac{1}{2}\frac{[(3\mu - N\rho H^2 n^2)(3\mu - 4N\rho H^2 n^2) - 18\nu^2 n^2]}{[(3\mu - N\rho H^2 n^2)^2 + (3\nu n)^2][(3\mu - 4N\rho H^2 n^2)^2 + 4(3\nu n)^2]} \cos 2n\tau \cdot \left(\frac{L_0}{\pi R^2}\right)^2 \\ & + \frac{1}{2}\frac{[(3\mu - N\rho H^2 n^2) 6\nu n + (3\mu - 4N\rho H^2 n^2) 3\nu n]}{[(3\mu - N\rho H^2 n^2)^2 + (3\nu n)^2][(3\mu - 4N\rho H^2 n^2)^2 + 4(3\nu n)^2]} \sin 2n\tau \cdot \left(\frac{L_0}{\pi R^2}\right)^2. \qquad (74) \end{aligned}$$

PART II.

Analytical Theorems and Observational Results.

1. *Comparison of Analytical and Experimental Results* (*Static Loading*).

It is now possible to turn to an examination of the question whether predictions that would analytically follow from the equation system developed in Part I are in accord with the results of observation and experiment. A close quantitative estimate can be reached for the plastic coefficient μ, but the precise ascertainment of the order of magnitude of the Rayleigh-Thompson viscous coefficient remains for further investigations, and its influence can in consequence only be traced qualitatively. Its magnitude provisionally appears to be of the order 2×10^6 C.G.S. units. Throughout the analytical development of the argument the assumption has been made that μ is effectively constant, and that it is of the order 2×10^9 C.G.S. units under plastico-viscous flow conditions, as contrasted with the value 4×10^{11} C.G.S. units for the rigidity coefficient under elastic conditions, that is with no permanent set imposed.

It would naturally be expected that the respective orders of magnitude would be different in the differing conditions, and that values would be lower in the plastic as contrasted with the elastic deformation. It is of considerable consequence to be assured that μ may be regarded as sensibly constant over the region of the large plastic strains imposed. Developing the analysis statically, that is with a balance at any instant between load and resistance, the deformation being assumed to take place extremely slowly, from (62)

$$F_{01} = \frac{L}{3\mu\pi R^2},$$

where R is the radius of the soft metal cylinder at the instant under consideration. The compression c is given by

$$\frac{c}{H} = F_{01}, \qquad (75)$$

where H is taken to be the height in the initial unstrained condition. Now the volume is assumed not to alter during compression, and with sufficiently slow pressing, and no friction at the flat surfaces, the shape remains cylindrical. Therefore, denoting the volume as V

$$\pi R^2 (H - c) = V. \qquad (76)$$

Combining (62) and (76)

$$L = 3\mu \frac{V}{H} \cdot \frac{c}{H - c}. \qquad (77)$$

Now the experimental results with which it is desired to make comparison have been carried out with cylinders previously compressed and subsequently heat treated. They do not start to compress immediately on application of load, since they retain a certain amount of residual strain. Moreover, the first movements cannot be regarded as strictly under static conditions, since the earlier compressions vary as the cube of the time for a uniform rate of increase of load. The point will be taken up in due course. It is during the later stages of compression, that is with the larger strains and heavier loads, that the conditions of pressing approach most closely to the conditions envisaged as static. The relevant experimental results will be quoted. They are to be found in (II), and have reference to a standard pressure-compression table known as the R G F static compression table, also to some special experiments carried out with a Buckton testing machine. The R G F static table shows that though compression starts on the application of a load equivalent to a pressure of approximately 1 ton/in.^{2} on a piston of 0·461 inch diameter, it is not until the pressure reaches a value approximating to 6 ton /in.^{2} that marked permanent set is experienced. To apply (77) it is necessary to assign a starting condition for permanent set, with a starting load, and as measurements are made from the initial standard condition this is taken to be c_0, with a load L_0.

From (77)

$$L_0 = 3\mu \frac{V}{H} \cdot \frac{c_0}{H - c_0}. \qquad (78)$$

Combining (77) and (78)

$$L - L_0 = 3\mu\pi R_0^2 \cdot \frac{\dfrac{c - c_0}{H - c_0}}{\left\{1 - \dfrac{c - c_0}{H - c_0}\right\}} \quad \dots\dots\dots\dots \quad (79)$$

where $R_0 = 0 \cdot 163$ inch, $H - c_0 = 0 \cdot 5$ inch, $c - c_0 =$ compression measured in inches.

Now if P be the pressure applied in ton/in.²

$$L - L_0 = (P - P_0) \frac{\pi}{4} (0 \cdot 461)^2, \quad \dots\dots\dots\dots \quad (80)$$

therefore

$$P - P_0 = \frac{3\mu}{2} \cdot \frac{\dfrac{c - c_0}{H - c_0}}{\left\{1 - \dfrac{c - c_0}{H - c_0}\right\}} \quad \dots\dots\dots\dots \quad (81)$$

There is this one formula available for testing the range of higher pressure-compression values, and at choice, as fitting constants, are P_0 and μ. Agreement cannot be expected for the low compressions, as will be indicated shortly, but for stepped compressions, assigning to P_0 the value 6, and to $3\mu/2$ the value 17 in the units used, values are given in Table I.

TABLE I.

Compression in inches.	Recorded pressure ton/in.².	Pressure from (81) ton/in.².	Difference ton/in.².
0·075	8·65	9·00	+0·35
0·100	10·35	10·25	—0·10
0·150	13·80	13·29	—0·51
0·200	17·75	17·33	—0·42
0·250	22·75	23·00	+0·25
0·275	26·00	26·78	+0·78

With the Buckton testing machine the values given in Table II are obtained.

TABLE II.

Recorded pressure ton/in.².	Pressure from (81) ton/in.².	Difference ton/in.².	Remaining height inches.	Diameter flats inches.	Diameter central inches.	Volume/π inch.³.
7·10	7·97	+0·87	0·4481	0·3376	0·3462	0·01321
10·30	10·29	—0·01	0·3992	0·3546	0·3690	0·01324
14·00	13·38	—0·62	0·3487	0·3683	0·3987	0·01318
16·90	16·84	—0·06	0·3052	0·3883	0·4285	0·01317
22·00	22·39	+0·39	0·2546	0·4282	0·4702	0·01327
29·00	29·80	+0·80	0·2081	0·4712	0·5213	0·01328
42·60	45·69	+3·39	0·1500	0·5625	0·6065	0·01315

In reading Table II it is to be recalled that columns 1, 4, 5, 6 are experimental values entered as recorded. The deformation is tabulated in terms of remaining height, not in terms of further compression ; the connection is made through

$$\text{Remaining height} + \text{Compression} = 0 \cdot 50 \text{ inch.}$$

The diameters measured are mean values and have been taken on the upper and lower flats, and equatorially. It is stated in the experimental record that the flat surfaces were dry. This would give a friction effect, increasing with increasing compression, but not unduly exaggerated. For a load of 42·6 ton/in.^{2}, and a compression from 0·50 inch to 0·150 of an inch the barrelling has attained a value not much exceeding some 10%. In this connection there is a further point of interest. It might be anticipated that with friction grip the system would register a higher resistance for a given compression, than would occur with no friction, as the system should be stiffer being more constrained. This is not so, the analogy is nearer to hydrodynamics than dynamics. With friction grip the upper surface normal stress acts over a relatively smaller area, so that the summed effect balanced against the load is less. This may possibly be part explanation of the difference noted in the last line of column 3. Column 2 has been calculated using the same values in (81) as were used for the corresponding entries in Table II. The values in column 7 have been calculated from the records entered in columns 4, 5, 6. They are exhibited in the form volume/π so that the percentage spread can be seen at a glance. They furnish sufficiently strong support for the assumption of incompressibility of the medium under plastic strain conditions. Axial cross-sectioning of the compressed cylinders shows that the boundary edges are sufficiently nearly circular in shape for this form to be taken in arriving at volumes enclosed.

Applying GULDIN's Theorem to the sector bounded by radii from the centre to the upper and lower extremes of the circular arc, and adding the volume of the upper and lower cones, not previously included, the total volume becomes

$$\frac{\pi}{3}(R_1^2 + 2R_0^2)\,H,$$

where R_1 is the radius of a flat, R_0 the equatorial radius, and H the height corresponding to the radii. The initial value of volume/π for the cylinders used is 0·01328 in.^{3}. Assigning to $3\mu/2$ the value 17 when the pressure is expressed in terms of ton/in.^{2} leads to the figure $1 \cdot 75 \times 10^9$ C.G.S. units for μ.

2. *Conditions at Start of Motion* (*Slow Rates of Compression*).

It is now necessary to examine the conditions at the start of slow compressing of a cylinder under an applied load increasing at a uniform rate. Some load is required to overcome the initial elastic resistance and residual plastic stress. Let this be denoted by A, and take the loading to be of the form

$$L(\tau) = A + B\tau,$$

where A and B are constants. As initial motions are mainly at issue it will suffice to take as the equation of motion

$$(3K + N\rho H^2 D^2)\, F_{01} = \frac{A + B\tau}{\pi R^2}. \qquad (82)$$

Then

$$F_{01} = I_1 e^{-\alpha_1\tau} + I_2 e^{-\alpha_2\tau} + \frac{1}{3K + N\rho H^2 D^2} \cdot \frac{A + B\tau}{\pi R^2}$$

$$= I_1 e^{-\alpha_1\tau} + I_2 e^{-\alpha_2\tau} + \frac{1}{3\mu}\left\{\left(A - \frac{\nu}{\mu} B\right) + B\tau\right\} \cdot \frac{1}{\pi R^2}, \qquad (83)$$

where α_1, α_2, are the roots of

$$\frac{N\rho H^2 \alpha^2}{3} - \nu\alpha + \mu = 0$$

$$\alpha_1 \to \frac{3\nu}{N\rho H^2}; \quad \alpha_2 \to \frac{\mu}{\nu}$$

$$DF_{01} = -\alpha_1 I_1 e^{-\alpha_1\tau} - \alpha_2 I_2 e^{-\alpha_2\tau} + \frac{B}{3\mu} \cdot \frac{1}{\pi R^2}. \qquad (84)$$

At

$$\tau = 0,\ F_{01} = \frac{A}{3\mu\pi R^2}, \quad DF_{01} = 0,$$

therefore

$$I_1 + I_2 + \frac{1}{3\mu}\left\{\left(A - \frac{\nu}{\mu} B\right)\right\} \frac{1}{\pi R^2} = \frac{A}{3\mu\pi R^2}$$

$$\alpha_1 I_1 + \alpha_2 I_2 - \frac{B}{3\mu} \cdot \frac{1}{\pi R^2} = 0,$$

or

$$I_1 + I_2 - \frac{\nu}{3\mu^2} \cdot \frac{B}{\pi R^2} = 0$$

$$\alpha_1 I_1 + \alpha_2 I_2 - \frac{\mu}{3\mu^2} \cdot \frac{B}{\pi R^2} = 0,$$

leading to

$$I_1 = -\frac{1}{9\mu^2} \cdot \frac{B}{\pi R^2} \cdot \frac{N\rho H^2 \alpha_2^2}{\alpha_1 - \alpha_2}; \quad I_2 = \frac{1}{9\mu^2} \cdot \frac{B}{\pi R^2} \cdot \frac{N\rho H^2 \alpha_1^2}{\alpha_1 - \alpha_2}. \qquad (85)$$

The observed further compression—C— at any instant τ is given by

$$\frac{C}{H} = F_{01} - \frac{A}{3\mu\pi R^2} = I_1 e^{-\alpha_1\tau} + I_2 e^{-\alpha_2\tau} - \frac{B}{3\mu^2}(\nu - \mu\tau) \frac{1}{\pi R^2}. \qquad (86)$$

For very extended times at low rates of load-increase the last term is the one of moment, and the expression on the right-hand side of (86) trends to the static formula previously considered. For intermediate times the relationship between (86) and the loading is complex. For short times, retaining only the lowest term surviving in the exponential expansions,

$$\frac{C}{H} = F_{01} - \frac{A}{3\mu\pi R^2} = \frac{1}{N\rho H^2} \cdot \frac{B}{\pi R^2} \cdot \frac{\tau^3}{\lfloor 3}.$$

Using

$$\tau = \frac{L(\tau) - A}{B} = \frac{(P - P_0)\,\pi R_p^{\,2}}{B},$$

where P_0 is the pressure at which compression starts, P is the pressure corresponding to the compression C, R_p is the radius of the piston on which the pressure acts,

$$(P - P_0)^3 = \frac{6}{\pi^2} \cdot \frac{N\rho HR^2}{R_p^{\,6}} \cdot B^2 \cdot C. \qquad (87)$$

From equation (87) it is at once seen that the pressure-compression curve for small compressions is concave to the compression axis, and that as the rate of loading—B—increases, other conditions remaining unchanged, the pressure corresponding to a definite compression increases. The inferences are in direct accord with observation. A pressure-compression curve obtained by pressing at a uniform rate of load-increase starts by being concave to the compression axis, passes through a point of inflexion, and ultimately becomes convex to the compression axis, as is required by (81). Corresponding to definite compression rates, pressure compression curves exhibit higher pressures for specific compressions as the rates are higher (II).

3. *Comparison of Analytical and Experimental Results* (*Dynamic Loading*).

So far comparison has been made between inferences from analysis and results of observation under conditions as nearly static as the operation of compressing will permit. Dynamic effects remain to be considered. Outstanding effects observed or deduced from observation under such conditions are :—

(i) With a load increasing and then decreasing a cylinder ceases to compress not at but after the instant of occurrence of maximum loading.

(ii) For a given maximum applied load the compression is greatest as conditions trend towards static.

(iii) For a given maximum applied load the compression becomes less and less as the rate of application is increased.

(iv) For a given maximum applied load as the piston mass is increased the compression becomes greater as the rate of application is increased. (The phenomenon of dynamic overshoot.)

(v) For a given maximum applied load and an exceptionally massive piston the compression becomes less as the rate of application is increased.

Conditions (iii), (iv), and (v) presume a load increasing and then decreasing.

The above statements may be accepted as theorems of the analysis. The investigation in the preceding paragraph shows that starting conditions are relatively of little consequence when high pressures and large strains are involved. To simplify the analysis, therefore, the medium will be taken to be unstrained and at rest when the loading is applied, and the form of $L(\tau)$ will be written as $L_0 \sin n\tau$. Second order effects follow qualitatively those of the first order. The analysis may consequently be restricted to the consideration of (*cf.* (74))

$$F_{01} = I_1 e^{-\alpha_1\tau} + I_2 e^{-\alpha_2\tau} + \frac{(3\mu - N\rho H^2 n^2)\sin n\tau - 3\nu n \cos n\tau}{(3\mu - N\rho H^2 n^2)^2 + (3\nu n)^2} \cdot \frac{L_0}{\pi R^2}.$$

Write

$$\frac{(3\mu - N\rho H^2 n^2)}{\sqrt{(3\mu - N\rho H^2 n^2)^2 + (3\nu n)^2}} = \cos\varepsilon, \quad \ldots \quad (88)$$

$$\frac{3\nu n}{\sqrt{(3\mu - N\rho H^2 n^2)^2 + (3\nu n)^2}} = \sin\varepsilon, \quad \ldots \quad (89)$$

$$\frac{L_0}{\pi R^2 \sqrt{(3\mu - N\rho H^2 n^2)^2 + (3\nu n)^2}} = C. \quad \ldots \quad (90)$$

The equations (88) and (89) repay attention. They show that when n is small ε is a small angle in the first quadrant. As n increases so does ε until

$$3\mu = N\rho H^2 n^2,$$

ε then has the value $\frac{1}{2}\pi$. For high values of n, $\cos\varepsilon$ becomes negative and the ultimate value of ε is π. There is thus no ambiguity connected with the value to be assigned to ε. Detailed consideration of equation (90) is deferred until more is known as to ranges of values for I_1 and I_2. From the expression for F_{01} by operating with D, and using (88), (89), (90)

$$-DF_{01} = \alpha_1 I_1 e^{-\alpha_1\tau} + \alpha_2 I_2 e^{-\alpha_2\tau} - nC\cos(n\tau - \varepsilon), \quad \ldots \quad (91)$$

$$F_{01} = I_1 e^{-\alpha_1\tau} + I_2 e^{-\alpha_2\tau} + C\sin(n\tau - \varepsilon). \quad \ldots \quad (92)$$

At the start of compression $F_{01} = 0$, $DF_{01} = 0$, giving, since $\tau = 0$,

$$I_1 + I_2 - C\sin\varepsilon = 0 \quad ; \quad \alpha_1 I_1 + \alpha_2 I_2 - nC\cos\varepsilon = 0$$

leading to

$$I_1 = \frac{n\cos\varepsilon - \alpha_2\sin\varepsilon}{\alpha_1 - \alpha_2} \quad ; \quad I_2 = \frac{\alpha_1\sin\varepsilon - n\cos\varepsilon}{\alpha_1 - \alpha_2},$$

or, using,

$$\frac{N\rho H^2\alpha^2}{3} - \nu\alpha + \mu = 0, \quad \ldots \quad (68)$$

$$I_1 = -\frac{n(n^2 + \alpha_2^2) N\rho H^2 C}{\sqrt{(3\mu - N\rho H^2 n^2)^2 + (3\nu n)^2}}, \quad \ldots \quad (93)$$

$$I_2 = +\frac{n(n^2 + \alpha_1^2) N\rho H^2 C}{\sqrt{(3\mu - N\rho H^2 n^2)^2 + (3\nu n)^2}}. \quad \ldots \quad (94)$$

Therefore I_1 is negative, I_2 is positive, and I_2 is $> | I_1 |$ since $\alpha_1 > \alpha_2$. Moreover, from the starting condition equations $\alpha_2 I_2 > | \alpha_1 I_1 |$ (cos ε positive) ; $\alpha_2 I_2 < | \alpha_1 I_1 |$ (cos ε negative). The end of compression is given by $DF_{01} = 0$; or by the value of τ ($\neq 0$) rendering

$$\cos (n\tau - \varepsilon) = \frac{\alpha_1 I_1}{nC} e^{-\alpha_1 \tau} + \frac{\alpha_2 I_2}{nC} e^{-\alpha_2 \tau} . \qquad (95)$$

Recasting the equation into the form

$$\cos (n\tau - \varepsilon) = \frac{\alpha_2 I_2}{nC} e^{-\alpha_2 \tau} \left\{ 1 + \frac{\alpha_1 I_1}{\alpha_2 I_2} e^{-(\alpha_1 - \alpha_2)\tau} \right\},$$

or, using (93), (94),

$$\cos (n\tau - \varepsilon) = \frac{\alpha_2 I_2}{nC} e^{-\alpha_2 \tau} \left\{ 1 - \frac{\alpha_1 (n^2 + \alpha_2^2)}{\alpha_2 (n^2 + \alpha_1^2)} e^{-(\alpha_1 - \alpha_2)\tau} \right\}, \qquad (96)$$

it is seen that the terms outside the bracket are always positive, whereas the bracket expression may become zero or negative for very large values of n. If the value of τ ($\neq 0$), satisfying the equation, satisfy this condition also then $\cos (n\tau - \varepsilon)$ will have a zero or negative value, or $n\tau > \frac{1}{2}\pi$ since ε is positive. The condition, therefore, that calls for detailed examination arises when the bracket expression is positive for the solution, for then $\cos (n\tau - \varepsilon)$ will have a finite positive value, and so $n\tau - \varepsilon < \frac{1}{2}\pi$, from which the inference cannot necessarily be drawn that $n\tau > \frac{1}{2}\pi$. Substituting in (96) from the previous equations, and using

$$\frac{3\mu}{N\rho H^2} = \alpha_1 \alpha_2 ; \qquad \frac{\mu}{\nu} = \frac{\alpha_1 \alpha_2}{\alpha_1 + \alpha_2} .$$

Then

$$\cos (n\tau - \varepsilon) = \frac{\alpha_2}{n} \cdot \frac{\alpha_1 \sin \varepsilon - n \cos \varepsilon}{\alpha_1 - \alpha_2} e^{-\alpha_2 \tau} \left\{ 1 - \frac{\alpha_1 (n^2 + \alpha_2^2)}{\alpha_2 (n^2 + \alpha_1^2)} e^{-(\alpha_1 - \alpha_2)\tau} \right\}$$

$$= \sin \varepsilon \cdot \frac{\alpha_2}{n} \cdot \frac{\alpha_1 - \frac{\mu}{\nu} - \frac{N\rho H^2 n^2}{3\nu}}{\alpha_1 - \alpha_2} e^{-\alpha_2 \tau} \left\{ 1 - \frac{\alpha_1 (n^2 + \alpha_2^2)}{\alpha_2 (n^2 + \alpha_1^2)} e^{-(\alpha_1 - \alpha_2)\tau} \right\}$$

$$= \sin \varepsilon \cdot \frac{\alpha_2}{n} \cdot \frac{\alpha_1^2 + n^2}{\alpha_1^2 - \alpha_2^2} e^{-\alpha_2 \tau} \left\{ 1 - \frac{\alpha_1 (n^2 + \alpha_2^2)}{\alpha_2 (n^2 + \alpha_1^2)} e^{-(\alpha_1 - \alpha_2)\tau} \right\}.$$

Observing that to ensure the bracket expression being positive, n must be kept relatively low, consider a range of values of n such that $n \ngtr 3\alpha_2$. The region covers most of the conditions likely to be experienced, for at the upper limit of n the time to maximum loading is given by

$$\tau = \frac{\pi}{6\alpha_2} = \frac{\pi}{6 \times 10^3} \rightarrow 5 \times 10^{-4} \text{ seconds.}$$

The range consequently includes times from the order mentioned upwards to conditions approaching static when n is very small. As mentioned previously $\alpha_1 \rightarrow 10^5$, $\alpha_2 \rightarrow 10^3$, $\mu \rightarrow 2 \times 10^9$, $\nu \rightarrow 2 \times 10^6$, n varies. Examining the bracket expression as $n \rightarrow 0$

$$\frac{\alpha_1 (n^2 + \alpha_2{}^2)}{\alpha_2 (n^2 + \alpha_1{}^2)} \rightarrow \frac{\alpha_2}{\alpha_1} \rightarrow 10^{-2},$$

and the exponential term is a fraction, since $\alpha_1 > \alpha_2$ and τ is positive. The expression is least when $\tau = 0$ and $\rightarrow$ unity as τ increases.

For $n = 3\alpha_2$

$$\frac{\alpha_1 (n^2 + \alpha_2{}^2)}{\alpha_2 (n^2 + \alpha_1{}^2)} = \frac{10\alpha_1\alpha_2}{\alpha_1{}^2 + 9\alpha_2{}^2} \rightarrow < \tfrac{1}{10}.$$

It is convenient to have the order of magnitude of the exponential term at the instant $\tau = \pi/2n$ for $n = 3\alpha_2$. In the conditions

$$e^{-(\alpha_1 - \alpha_2)\tau} = e^{-(10^5 - 10^3)\frac{\pi}{6 \times 10^3}} = e^{-\frac{33\pi}{2}}.$$

Thus the bracket expression again trends to unity as τ increases. Effectively therefore the equation to be solved may be regarded as requiring the ascertainment of the points of intersection of two curves given to a time basis by the expressions on the left- and right-hand sides of the equation, the bracket expression being a fraction very closely approximating to unity. From the starting conditions the curves intersect when $\tau = 0$. As τ increases the cosine expression increases to unity, and then decreases, ultimately becoming negative. The exponential expression on the right-hand side falls steadily in value as τ increases, but the right-hand side never becomes negative in value. The two curves will consequently intersect at some point as τ increases, giving the value of τ required. If therefore it can be shown that at the time instant given by $n\tau = \frac{1}{2}\pi$ the ordinate of the second curve is less in value than the ordinate of the cosine curve, both ordinates being positive, then the point of intersection must lie beyond the value of τ given by $n\tau = \frac{1}{2}\pi$, that is the motion ceases after the instant of occurrence of maximum loading.

At $\tau = \pi/2n$, $\cos (n\tau - \varepsilon) = \sin \varepsilon$.

At $\tau = \pi/2n$ the expression on the right-hand side becomes

$$\sin \varepsilon \, . \, \frac{\alpha_2}{n} \, . \, \frac{\alpha_1{}^2 + n^2}{\alpha_1{}^2 - \alpha_2{}^2} \, e^{-\frac{\pi}{2}\frac{\alpha_2}{n}} \left\{ 1 - \frac{\alpha_1 (n^2 + \alpha_2{}^2)}{\alpha_2 (n^2 + \alpha_1{}^2)} \, e^{-(\alpha_1 - \alpha_2)\frac{\pi}{2n}} \right\}.$$

Now $\dfrac{\alpha_1{}^2 + n^2}{\alpha_1{}^2 - \alpha_2{}^2} \rightarrow (1 + 10^{-4})$ (n small); $\rightarrow (1 + 10^{-3})$ $(n = 3\alpha_2)$ whilst the bracket expression $\rightarrow 1$. Moreover, $xe^{-\frac{\pi}{2}x}$ (x positive) is $\not> 2/\pi e$ (of the order 1/4). Sufficiently accurately therefore the equation to be solved to give the instant of occurrence of rest is

$$\cos (n\tau - \varepsilon) = \sin \varepsilon \, . \, \frac{\alpha_2}{n} \, e^{-\alpha_2 \tau}.$$

Write $n\tau = \frac{\pi}{2} + \phi$, where, by the previous argument, ϕ is positive. Therefore

$$\phi = \varepsilon - \sin^{-1}\left(\sin\varepsilon\,\frac{\alpha_2}{n}\,.\,e^{-\frac{\alpha_2}{n}\cdot\frac{\pi}{2}}\,.\,e^{-\frac{\alpha_2}{n}\phi}\right). \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (97)$$

Now

$$\sin^{-1} z = z + \tfrac{1}{2}\frac{z^3}{3} + \frac{1\,.\,3}{2\,.\,4}\frac{z^5}{5} + \ldots \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (98)$$

Regarding z as the expression within the bracket in (97); $\sin\varepsilon < 1$, $\frac{\alpha_2}{n}e^{-\frac{\alpha_2}{n}\frac{\pi}{2}} < \frac{1}{4}$, $e^{-\frac{\alpha_2}{n}\phi}$ is a fraction, since ϕ is positive. At the most, therefore, z cannot be greater than $\frac{1}{4}$. The second term on the right in (98) is not greater than 1/384. Neglect of this term will result in an error in ϕ of the amount shown; but as τ is ascertained from

$$\tau = \frac{\pi}{2n} + \frac{\phi}{n},$$

and, as n is now to be regarded as possessing values $\not< 10^2$, the error in τ introduced by replacing equation (97) using only the first term as shown in the expansion (98) is negligible. Rewrite equation (97), therefore, as

$$\phi = \varepsilon - \sin\varepsilon\,\frac{\alpha_2}{n}\,.\,e^{-\frac{\alpha_2}{n}\frac{\pi}{2}}\,.\,e^{-\frac{\alpha_2}{n}\phi}, \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \qquad (99)$$

and apply a LAPLACE expansion. There results

$$\phi = \varepsilon - \frac{n}{\alpha_2}\left\{\frac{\frac{\alpha_2}{n}\sin\varepsilon}{e^{\frac{\alpha_2}{n}\varepsilon}}\cdot\frac{\frac{\alpha_2}{n}}{e^{\frac{\alpha_2}{n}\frac{\pi}{2}}} + \frac{2}{\underline{|2}}\left(\frac{\frac{\alpha_2}{n}\sin\varepsilon}{e^{\frac{\alpha_2}{n}\varepsilon}}\cdot\frac{\frac{\alpha_2}{n}}{e^{\frac{\alpha_2}{n}\frac{\pi}{2}}}\right)^2 + \frac{3^2}{\underline{|3}}\left(\ldots\right)^3 + \frac{4^3}{\underline{|4}}\left(\ldots\right)^4 + \ldots\right\},$$

$$. \quad . \quad . \quad . \quad . \qquad (100)$$

provided the expansion is convergent. This follows:

since $\sin\varepsilon < \varepsilon$, $\frac{\frac{\alpha_2}{n}\sin\varepsilon}{e^{\frac{\alpha_2}{n}\varepsilon}} < \frac{1}{e}$, also $\frac{\frac{\alpha_2}{n}}{e^{\frac{\alpha_2}{n}\cdot\frac{\pi}{2}}} < \frac{2}{\pi e}$ (in general).

Further

$$\underset{n\to\infty}{\text{Lt}}\frac{(n+1)^n}{\underline{|n+1}} : \frac{n^{n-1}}{\underline{|n}} = \underset{n\to\infty}{\text{Lt}}\left(1+\frac{1}{n}\right)^n = e.$$

Now

$$\sin\varepsilon = \frac{n(\alpha_1+\alpha_2)}{\sqrt{n^2+\alpha_1^2}\,\sqrt{n^2+\alpha_2^2}} = \frac{3\nu n}{\sqrt{(3\mu - N\rho H^2 n^2)^2 + (3\nu n)^2}},$$

(from (89)), and it is of interest to observe that the direct cause of lag is viscosity. At low speeds of loading the time interval from start of compression to the occurrence of rest is given by

$$\tau = \frac{\pi}{2n} + \frac{\varepsilon}{n} = \frac{\pi}{2n} + \frac{\nu}{\mu},$$

the measure of the interval—instant of occurrence of maximum loading to instant of occurrence of rest—being the ratio of the RAYLEIGH-THOMPSON viscosity coefficient to the plastic coefficient.

It is now possible to examine qualitatively the indications of the relationship (90). At the instant of occurrence of rest

$$F_{01} = I_1 e^{-\alpha_1 \tau} + I_2 e^{-\alpha_2 \tau} + C \sin (n\tau - \varepsilon), \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad . \quad (92)$$

where the constants of integration, the exponential terms, and the trigonometric term are subject to the restrictions discussed previously. Since

$$\frac{\alpha_1 I_1 e^{-\alpha_1 \tau}}{nC}, \qquad \frac{\alpha_2 I_2 e^{-\alpha_2 \tau}}{nC},$$

have been shown to be relatively small order quantities, the terms $I_1 e^{-\alpha_1 \tau}$, $I_2 e^{-\alpha_2 \tau}$ are comparatively negligible in respect to $C \sin (n\tau - \varepsilon)$, the latter factor corresponding to an angle in the neighbourhood of $\frac{1}{2}\pi$. The expression $C \sin (n\tau - \varepsilon)$ may, therefore, be taken as a measure of the compression, or writing it more explicitly

$$C \sin (n\tau - \varepsilon) = \frac{L_0}{\pi R^2} \cdot \frac{1}{\sqrt{(3\mu - N\rho H^2 n^2)^2 + (3\nu n)^2}} \sin (\tfrac{1}{2}\pi - \phi_n),$$

where ϕ_n is a small angle, for n small, gradually increasing as n increases. The significance of the expression under the square-root sign comes more clearly to light when the expression is re-written

$$\frac{L_0}{\pi R^2} \frac{1}{\sqrt{9\mu^2 + (3\nu n)^2 - N\rho H^2 n^2 (6\mu - N\rho H^2 n^2)}} \sin (\tfrac{1}{2}\pi - \phi_n).$$

Theorem—

(i) has been demonstrated;

(ii) time to maximum loading extended: piston very light; conditions indicated by $n \to 0$, $N \to 0$, $\phi_n \to 0$, $\sin (\frac{1}{2}\pi - \phi_n) \to 1$;

$$\text{expression} \to \frac{L_0}{\pi R^2} \cdot \frac{1}{3\mu} \cdot \text{(the static value and the greatest).}$$

(iii) Time to maximum loading decreasing; piston very light; conditions indicated by $n \to$ increase, $N \to 0$, $\phi_n \to$ increase, $\sin (\frac{1}{2}\pi - \phi_n) < 1$;

$$\text{expression} \to < \frac{L_0}{\pi R^2} \cdot \frac{1}{\sqrt{9\mu^2 + (3\nu n)^2}}$$

The compression diminishes from two causes, decrease in numerator increase in denominator.

(iv) Time to maximum loading remaining the same as in (iii); piston becoming more massive but not to such an extent as to render negative $(6\mu - N\rho H^2 n^2)$, conditions indicated by n as in (iii), $N \rightarrow$ increase, $\sin(\frac{1}{2}\pi - \phi_n) < 1$;

$$\text{expression} \rightarrow \frac{L_0}{\pi R^2} \frac{1}{\sqrt{9\mu^2 + (3\nu n)^2 - (\text{a positive quantity})}}.$$

The compression tends to increase again, illustrating the phenomenon of dynamic overshoot.

(v) Time to maximum loading extremely small; piston so massive that the expression $(6\mu - N\rho H^2 n^2)$ is negative; conditions $n \rightarrow$ large, $N \rightarrow$ large,

$$\sin(\tfrac{1}{2}\pi - \phi_n) \rightarrow 0;$$

$$\text{expression} \rightarrow < \frac{L_0}{\pi R^2} \frac{1}{\sqrt{9\mu^2 + (3\nu n^2) + (\text{a positive quantity})}}.$$

The overshoot gradually disappears and becomes undershoot.

The last condition is extreme, and indicates that with an exceedingly massive piston and an extremely short time of load application the compression will scarcely be appreciable. Such a condition is highly undesirable; and is usually avoided.

4. *Hardness and Rate of Compression.*

One matter having a metallurgical bearing remains for note. Hardness not only increases with stress but, other things being equal, also with increased rate of application of load. For since $K = \mu + \nu D$ the D effect on F_{01} is always positive, additive to the μ effect, and greater as the velocity of compression is greater.

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Summary.

Subject to the restriction customarily imposed of an order of strain smallness when stress-strain relationships are being considered an analysis is developed for the motion of a soft metal cylinder when subjected to a crushing load directed axially, surface friction being regarded as absent. The metal is considered to be homogeneous, isotropic, and incompressible, and coefficients of plasticity and viscosity are used in the stress-strain relationships. The modifying influence of rate of application of load is examined. Theorems of the analysis are shown to be in accord with general conclusions drawn from experiment.